Understanding Pure Mathematics

(Sadler & Thorning)

Chapter 20: Calculus IV: Further integration

Exercise 20A

(1) $I = \int 6x \sin(x^2-4)\, dx$

$u = x^2 - 4$, so $du = 2x\, dx$

$\therefore\ 3\, du = 6x\, dx$

$\therefore\ I = \int 3 \cdot \sin u\, du = -3\cos u + c$

$= -3\cos(x^2-4) + c$

(2) $I = \int 5x \cos(5-x^2)\, dx$

let $u = 5 - x^2$, so $du = -2x\, dx$

$\therefore\ -\frac{5}{2}\, du = 5x\, dx$

hence $I = \int -\frac{5}{2}\cos u\, du = -\frac{5}{2}\sin u + c$

$= -\frac{5}{2}\sin(5-x^2) + c$

(3) $I = \int 3x\sqrt{1+x^2}\, dx$

let $u = 1+x^2$, so $du = 2x\, dx$

$\therefore \quad \frac{3}{2} du = 3x\, dx$

hence $I = \int \frac{3}{2}\sqrt{u}\, du = \int \frac{3}{2} u^{1/2}\, du$

$= \frac{3}{2}\left(\frac{2}{3}\right) u^{3/2} + c$

$= (1+x^2)^{3/2} + c$

(4) $I = \int 3x\,(x^2+6)^5\, dx$

let $u = x^2+6$, so $du = 2x\, dx$

$\therefore \quad \frac{3}{2} du = 3x\, dx$

$\therefore I = \int \frac{3}{2} u^5\, du = \frac{3}{2}\cdot\frac{1}{6} u^6 + c$

$= \frac{1}{4}(x^2+6)^6 + c$

(5) $I = \int x(x+2)^9\, dx$

let $u = x+2$, so $du = dx$

and $u - 2 = x$

$\therefore \quad I = \int (u-2)\, u^9 \, du$

$= \int u^{10} - 2u^9 \, du$

$= \frac{1}{11} u^{11} - \frac{2}{10} u^{10} + c$

$= \frac{1}{11} (x+2)^{11} - \frac{1}{5} (x+2)^{10} + c$

⑥ $I = \int 5x^2 (x-3)^8 \, dx$

let $u = x - 3$, so $du = dx$

and $u + 3 = x$

$\therefore \quad I = \int 5 (u+3)^2 \cdot u^8 \, du$

$= \int (5u^2 + 30u + 45) \cdot u^8 \, du$

$= \int 5u^{10} + 30u^9 + 45u^8 \, du$

$= \frac{5}{11} u^{11} + \frac{30}{10} u^{10} + \frac{45}{9} u^9 + c$

$= \frac{5}{11} (x-3)^{11} + 3 (x-3)^{10} + 5 (x-3)^9 + c$

(7) $I = \int 9x\,(3x+2)^3\,dx$

Let $u = 3x+2$, so $du = 3\,dx$

$\therefore\ 3\,du = 9\,dx$

Also $\frac{u-2}{3} = x$

Hence

$$I = \int 3\left(\frac{u-2}{3}\right) \cdot u^3\,du$$

$$= \int (u-2) \cdot u^3\,du$$

$$= \int u^4 - 2u^3\,du$$

$$= \frac{u^5}{5} - \frac{2u^4}{4} + c$$

$$= \frac{1}{5}(3x+2)^5 - \frac{1}{2}(3x+2)^4 + c$$

(8) $I = \int 7x\,(2x+3)^5$

Let $u = 2x+3$, so $du = 2dx$

$\therefore\ \frac{7}{2}\,du = 7\,dx$

and Also $\frac{u-3}{2} = x$

$$\therefore \quad I = \int \frac{7}{2}\left(\frac{u-3}{2}\right) \cdot u^5 \, du$$

$$= \int \left(\frac{7}{4}u - \frac{21}{4}\right) \cdot u^5 \, du$$

$$= \int \frac{7}{4}u^6 - \frac{21}{4}u^5 \, du$$

$$= \frac{1}{4}u^7 - \frac{21}{24}u^6 + c$$

$$= \frac{1}{4}(2x+3)^7 - \frac{7}{8}(2x+3)^6 + c$$

(9) $I = \int \frac{3x}{\sqrt{2x+3}} \, dx = \int 3x \cdot (2x+3)^{-\frac{1}{2}} \, dx$

let $u = 2x+3$, so $du = 2\,dx$

$$\therefore \quad \frac{3}{2}\,du = 3\,dx$$

and also $\frac{u-3}{2} = x$

$$\therefore \quad I = \int \frac{3}{2}\left(\frac{u-3}{2}\right) \cdot u^{-\frac{1}{2}} \, du$$

$$= \int \left(\frac{3}{4}u - \frac{9}{4}\right) \cdot u^{-\frac{1}{2}} \, du$$

$$= \int \frac{3}{4}u^{\frac{1}{2}} - \frac{9}{4}u^{-\frac{1}{2}} \, du$$

$$= \frac{3}{4}\left(\frac{2}{3}\right) u^{3/2} - \frac{9}{4}(2)\, u^{1/2} + c$$

$$= \frac{1}{2} u^{3/2} - \frac{9}{2} u^{1/2} + c$$

$$= \frac{1}{2}(2x+3)^{3/2} - \frac{9}{2}(2x+3)^{1/2} + c$$

(10) $I = \int \frac{1}{\sqrt{1-x^2}}\, dx$

Let $x = \sin u$, so $dx = \cos u \, du$

$$\therefore \quad I = \int \frac{1}{\sqrt{1-\sin^2 u}} \cdot \cos u \, du$$

$$= \int \frac{1}{\sqrt{\cos^2 u}} \cdot \cos u \, du$$

$$= \int \frac{\cos u}{\cos u}\, du = u + c$$

But $x = \sin u$, so $u = \sin^{-1} x$

$$\therefore \quad I = \sin^{-1} x + c$$

(11) $I = \int \frac{1}{4+9x^2}\, dx$

Let $x = \frac{2}{3}\tan u$, so $dx = \frac{2}{3}\sec^2 u\, du$

$\therefore\ I = \int \frac{1}{4+9\left(\frac{4}{9}\tan^2 u\right)} \cdot \frac{2}{3}\sec^2 u\, du$

$= \int \frac{1}{4+4\tan^2 u} \cdot \frac{2}{3}\sec^2 u\, du$

$= \frac{2}{3}\int \frac{\sec^2 u}{4\sec^2 u}\, du$

$= \frac{1}{6}\int 1\, du = \frac{1}{6}u + c$

But $x = \frac{2}{3}\tan u$ So $u = \tan^{-1}\frac{3x}{2}$

$\therefore\ I = \frac{1}{6}\cdot\tan^{-1}\frac{3x}{2} + c$

(12) $I = \int \frac{1}{\sqrt{25-4x^2}}\, dx$

Let $x = \frac{5}{2}\sin u$, So $dx = \frac{5}{2}\cos u\, du$

$\therefore\ I = \int \frac{1}{\sqrt{25-4\left(\frac{25}{4}\sin^2 u\right)}} \cdot \frac{5}{2}\cos u\, du$

$$I = \int \frac{1}{\sqrt{25 - 25\sin^2 u}} \cdot \frac{5}{2}\cos u \; du$$

$$= \int \frac{1}{\sqrt{25\cos^2 u}} \cdot \frac{5}{2}\cos u \; du$$

$$= \frac{5}{2} \cdot \frac{1}{5} \int \frac{\cos u}{\cos u} \; du = \frac{1}{2}\int 1 \; du$$

$$= \frac{1}{2} u + c$$

But $x = \frac{5}{2}\sin u$, so $u = \sin^{-1} \frac{2x}{5}$

$$\therefore \quad I = \frac{1}{2}\sin^{-1}\frac{2x}{5} + c$$

13 $\quad I = \int \frac{1}{\sqrt{4 - 9x^2}} \; dx$

Let $x = \frac{2}{3}\sin u$, so $dx = \frac{2}{3}\cos u \; du$

$$\therefore \quad I = \int \frac{1}{\sqrt{4 - 9\left(\frac{4}{9}\sin^2 u\right)}} \cdot \frac{2}{3}\cos u \cdot du$$

$$= \int \frac{1}{\sqrt{4 - 4\sin^2 u}} \cdot \frac{2}{3}\cos u \; du$$

$$= \frac{2}{3}\int \frac{1}{\sqrt{4\cos^2 u}} \cdot \cos u \; du$$

$$= \frac{1}{3}\int \frac{\cos u}{\cos u} \; du = \frac{1}{3} u + c = \frac{1}{3}\sin^{-1}\frac{3x}{2} + c$$

(14) $I = \int \frac{1}{4+5x^2}\, dx$

Let $x = \frac{2}{\sqrt{5}} \tan u$, so $dx = \frac{2}{\sqrt{5}} \sec^2 u \, du$

$\therefore\ I = \int \frac{1}{4+5\left(\frac{4}{5}\tan^2 x\right)} \cdot \frac{2}{\sqrt{5}} \sec^2 u \, du$

$= \int \frac{1}{4+4\tan^2 x} \cdot \frac{2}{\sqrt{5}} \sec^2 u \, du$

$= \int \frac{2}{\sqrt{5}} \frac{1}{4\sec^2 u} \cdot \sec^2 u \, du$

$= \frac{2}{4\sqrt{5}} \int 1 \, du = \frac{1}{2\sqrt{5}} u + c$

But $x = \frac{2}{\sqrt{5}} \tan u$, so $u = \tan^{-1} \frac{5x}{\sqrt{2}}$

Hence $I = \frac{1}{2\sqrt{5}} \tan^{-1} \frac{5x}{\sqrt{2}} + c$

(15) $$I = \int \sqrt{25 - x^2} \cdot dx$$

Let $x = 5\sin\theta$, so $dx = 5\cos\theta \, d\theta$

$$\therefore \quad I = \int \sqrt{25 - 25\sin^2\theta} \cdot 5\cos\theta \, d\theta$$

$$= \int \sqrt{25\cos^2\theta} \cdot 5\cos\theta \, d\theta$$

$$= 25 \int \cos^2\theta \, d\theta$$

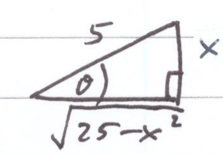
By the trig identity $\cos 2\theta = 2\cos^2\theta - 1$ we have $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$.

$$\therefore \quad I = 25 \int \tfrac{1}{2}(\cos 2\theta + 1) \, d\theta$$

$$= \frac{25}{2}\left[\tfrac{1}{2}\sin 2\theta + \theta\right] + c$$

$$= \frac{25}{2}\left(\sin\theta\cos\theta + \theta\right) + c$$

But $x = 5\sin\theta$, so $\sin\theta = \dfrac{x}{5}$



& $\cos\theta = \dfrac{\sqrt{25 - x^2}}{5}$

So $$I = \frac{25}{2}\left(\frac{x}{5} \cdot \frac{\sqrt{25 - x^2}}{5} + \sin^{-1}\frac{x}{5}\right) + c$$

(16) $$I = \int \sqrt{1-4x^2}\,dx$$

Let $x = \frac{1}{2}\sin\theta$, so $dx = \frac{1}{2}\cos\theta\,d\theta$

So
$$I = \int \sqrt{1-4\left(\frac{1}{4}\sin^2\theta\right)} \cdot \frac{1}{2}\cos\theta\,d\theta$$

$$= \frac{1}{2}\int \sqrt{1-\sin^2\theta} \cdot \cos\theta\,d\theta$$

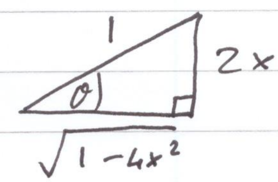
$$= \frac{1}{2}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{1}{2}(\cos 2\theta + 1)\,d\theta,$$

from $\cos 2\theta = 2\cos^2\theta - 1$.

hence $$I = \frac{1}{4}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c$$

But $x = \frac{1}{2}\sin\theta$, so $\frac{2x}{1} = \sin\theta$



so $\cos\theta = \sqrt{1-4x^2}$

$$\therefore\ I = \frac{1}{4}\left(\sin\theta\cos\theta + \theta\right) + c$$

$$= \frac{1}{4}\left(2x \cdot \sqrt{1-4x^2} + \sin^{-1} 2x\right) + c$$

(17) (a) $I = \int \frac{1}{1+16x^2}\,dx$

Let $u = 4x$, so $du = 4\,dx \Rightarrow \frac{1}{4}\,du = dx$

$\therefore \quad I = \int \frac{1}{1+u^2} \cdot \frac{1}{4}\,du$

$= \frac{1}{4}\int \frac{1}{1+u^2}\,du = \frac{1}{4}\tan^{-1} u + c$

$= \frac{1}{4}\tan^{-1}(4x) + c$

(b) $I = \int \frac{1}{\sqrt{9-16x^2}}\,dx$

Let $u = 4x$, so $du = 4\,dx \Rightarrow \frac{1}{4}\,du = dx$

So $I = \int \frac{1}{\sqrt{9-u^2}} \cdot \frac{1}{4}\,du$

$= \frac{1}{4}\sin^{-1}\left(\frac{u}{3}\right) + c$

$= \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + c$

(c) $I = \int \frac{1}{4+5x^2}\,dx$

Let $u = x\sqrt{5}$, so $du = \sqrt{5}\,dx \Rightarrow \frac{1}{\sqrt{5}}\,du = dx$

$\therefore \quad I = \frac{1}{\sqrt{5}} \int \frac{1}{4+u^2}\,du$

$= \frac{1}{\sqrt{5}} \cdot \frac{1}{2} \int \frac{2}{4+u^2}\,du$

(Multiply & divide by 2 so that we can use the formula correctly)

$\therefore \quad I = \frac{1}{2\sqrt{5}} \cdot \tan^{-1}\left(\frac{x}{2}\right) + c$

(d) $I = \int \frac{1}{\sqrt{5-2x^2}} \cdot dx$

Let $u = x\sqrt{2}$, so $du = \sqrt{2}\,dx \Rightarrow \frac{1}{\sqrt{2}}\,du = dx$

$\therefore \quad I = \int \frac{1}{\sqrt{5-u^2}} \cdot \frac{1}{\sqrt{2}}\,du$

$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$

(18) (a) $I = \int \dfrac{3}{x^2 - 2x + 10}\, dx$

$= \int \dfrac{3}{x^2 - 2x + 1 + 9}\, dx$

$= \int \dfrac{3}{(x-1)^2 + 9}\, dx$

let $3u = (x-1)$, so $3\,du = dx$

so $I = \int \dfrac{3}{9u^2 + 9} \cdot 3\, du$

$= \int \dfrac{1}{u^2 + 1}\, du$

$= \tan^{-1} u + c$

$= \tan^{-1} \dfrac{(x-1)}{3} + c$

(b) $I = \int \frac{1}{x^2+6x+13}\,dx$

$= \int \frac{1}{x^2+6x+9+4}\,dx$

$= \int \frac{1}{(x+3)^2+4}\,dx$

Let $2u = (x+3)$, so $2\,du = dx$

$\therefore\ I = \int \frac{1}{4u^2+4} \cdot 2\,du$

$= \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1} u + c$

$= \frac{1}{2}\tan^{-1}\frac{(x+3)}{2} + c$

(c) $I = \int \frac{3}{4x^2-12x+13}\,dx$

$= \int \frac{3}{(2x-3)^2+4}\,dx$

Let $2u = (2x-3)$, so $2\,du = 2.dx$

$\therefore\ du = dx$

Hence $I = \int \frac{3}{4u^2+4} \cdot du$

$$= \frac{3}{4} \int \frac{1}{u^2+1} \, du$$

$$= \frac{3}{4} \tan^{-1} u + c$$

$$= \frac{3}{4} \tan^{-1} \left(\frac{2x-3}{2}\right) + c$$

(19) $I = \int (x+2)(2x-3)^6 \, dx$

The power "6" on $(2x-3)$ is the "problem". So let $u = 2x-3$

& we will have u^6 which is much easier to handle.

So let $u = 2x-3$, $\therefore$ $du = 2\,dx \Rightarrow \frac{1}{2} du = dx$

and $x = \frac{u+3}{2}$

$$\therefore I = \int \left(\frac{u+3}{2} + 2\right) u^6 \cdot \frac{1}{2} \, du$$

$$= \frac{1}{2} \int \frac{u^7}{2} + 3\tfrac{1}{2} u^6 \, du = \frac{1}{2}\left(\frac{1}{8}\frac{u^8}{2} + \frac{7}{2} \cdot \frac{1}{7} u^7\right) + c$$

$$= \frac{1}{32} \cdot (2x-3)^8 + \frac{1}{4}(2x-3)^7 + c$$

$$= \frac{1}{32}(2x-3)^7(2x-3) + \frac{8}{32}(2x-3)^7 + c$$

$$= \frac{1}{32}(2x-3)^7\left[2x-3+8\right]$$

$$= \frac{1}{32}(2x-3)^7(2x+5)$$

(20) $I = \int x \cdot \sqrt{2x+1}\, dx$

Always try to get Rid of Square Roots. We do This by Setting up a Squared function which will cancel The Square Root.

$\therefore$ let $u^2 = 2x+1$; $\therefore \underbrace{2u\,du = 2\,dx}_{\text{Implicit diff.}}$

$\therefore x = \frac{u^2-1}{2}$

So $I = \int \frac{u^2-1}{2} \cdot \sqrt{u^2} \cdot u\,du$

$$= \int \left(\tfrac{1}{2}u^2 - \tfrac{1}{2}\right) \cdot u^2\, du$$

$$= \int \tfrac{1}{2}u^4 - \tfrac{1}{2}u^2\, du = \frac{1}{2}\frac{u^5}{5} - \frac{1}{2}\frac{u^3}{3} + c$$

$$= \frac{1}{10}\sqrt{(2x+1)^5} - \frac{1}{6}\sqrt{(2x+1)^3} + c$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + c$$

$$= (2x+1)^{3/2}\left[\frac{1}{10}(2x+1) - \frac{1}{6}\right]$$

$$= (2x+1)^{3/2}\left[\frac{6(2x+1)-10}{60}\right]$$

$$= \frac{1}{15}(3x-1)(2x+1)^{3/2}$$

on This occasion we could also have used $u = 2x+1$ to get

$I = \frac{1}{2}\int \frac{u-1}{2}\cdot\sqrt{u}\ du$ and This would have given The Same Result. See (21).

(21) $I = \int \frac{x}{\sqrt{2x+1}}\,dx$

let $u = 2x+1 \rightarrow du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

$$\frac{u-1}{2} = x$$

$$\therefore I = \frac{1}{2}\int (u-1)\cdot\frac{1}{\sqrt{u}}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{4}\int (u-1)\,u^{-1/2}\,du = \frac{1}{4}\int u^{1/2} - u^{-1/2}\,du$$

$$= \frac{1}{4}\left[\frac{2}{3} u^{3/2} - 2u^{1/2} \right] + c$$

$$= \frac{1}{4}\left(\frac{2}{3}(2x+1)^{3/2} - 2(2x+1)^{1/2} \right) + c$$

$$= (2x+1)^{1/2}\left[\frac{1}{6}(2x+1) - \frac{1}{2} \right] + c$$

$$= \sqrt{2x+1}\left[\frac{2(2x+1) - 6}{12} \right] + c$$

$$= \frac{1}{3}(x-1)\sqrt{2x+1}$$

(22) $I = \displaystyle\int \frac{1}{\sqrt{25-4x^2}}\,dx$

Consider $\cos^2\theta + \sin^2\theta = 1$. From this we can get

$\cos\theta = \sqrt{1-\sin^2\theta}$. The structure of the RHS is $\sqrt{\square - \square^2}$

which is the same as the structure of the denominator of I.

Because of this we will use a sin or cos substitution.

But it would be nice if the denom was $\sqrt{25-25x^2}$ so that we could use the identity $25\cos^2 x + 25\sin^2 x = 25$.

So pick our substitution so as to get a $25x^2$ term

$\therefore$ let $x = \frac{5}{2}\sin\theta$, $\therefore$ $x^2 = \frac{25}{4}\sin^2\theta$

$dx = \frac{5}{2}\cos\theta\, d\theta$

$$\therefore \quad I = \int \frac{1}{\sqrt{25 - 4\left(\frac{25}{4}\sin^2\theta\right)}} \cdot \frac{5}{2}\cos\theta \, d\theta$$

$$= \frac{5}{2}\int \frac{1}{\sqrt{25 - 25\sin^2\theta}} \cdot \cos\theta \, d\theta$$

$$= \frac{5}{2}\int \frac{1}{\sqrt{25\cos^2\theta}} \cdot \cos\theta \, d\theta$$

$$= \frac{5}{2}\cdot\frac{1}{5}\int 1 \, d\theta = \frac{1}{2}\theta + c$$

$$= \frac{1}{2}\sin^{-1}\frac{2x}{5} + c$$

we could have used $x = \frac{5}{2}\cos\theta$ also. The Result would have looked different but would still have been correct.

(23) $\int \sqrt{9 - x^2} \cdot dx$.

Let $x = 3\sin\theta$. (See (22) for Explanation as to how to choose this substitution)

$dx = 3\cos\theta \, d\theta$

So $I = \int \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta \, d\theta$

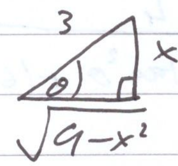
$$= \int \sqrt{9\cos^2\theta} \cdot 3\cos\theta \, d\theta = \int 9\cos^2\theta \, d\theta$$

By $\cos 2\theta = 2\cos^2\theta - 1$ we have $\dfrac{\cos 2\theta + 1}{2} = \cos^2\theta$

$$\therefore \quad I = \frac{9}{2}\int \cos 2\theta + 1 \, d\theta$$

$$= \frac{9}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c$$

$$= \frac{9}{4}\left(2\sin\theta\cos\theta\right) + \frac{9}{2}\theta + c$$



But $x = 3\sin\theta \Rightarrow \sin\theta = \dfrac{x}{3}$

& $\cos\theta = \frac{1}{3}\sqrt{9-x^2}$

So $$I = \frac{9}{2}\left(\frac{x}{3}\cdot\frac{1}{3}\sqrt{9-x^2}\right) + \frac{9}{2}\sin^{-1}\frac{x}{3} + c$$

$$= \frac{1}{2}\left[x\sqrt{9-x^2} + 9\sin^{-1}\frac{x}{3}\right] + c$$

24 $$I = \int \frac{4}{x^2+2x+17}\,dx$$

$$= \int \frac{4}{x^2+2x+1+16}\,dx$$

$$= \int \frac{4}{(x+1)^2+4^2}\,dx$$

Notice that $1 + \tan^2\theta = \sec^2\theta$ has the structure $\square^2 + \square^2$ on the LHS.
This is the same structure as the denominator of I.

What would be nice is if we had $(4\tan\theta)^2 + 4^2$ in the denom.

So let $x+1 = 4\tan\theta$, $\therefore (x+1)^2 = 16\tan^2\theta$

$$dx = 4\sec^2\theta \, d\theta$$

So $$I = \int \frac{4}{16\tan^2\theta + 16} \cdot 4\sec^2\theta \, d\theta$$

$$= \int \frac{1}{\sec^2\theta} \cdot \sec^2\theta \, d\theta = \int 1 \, d\theta$$

$$= \theta + c$$

$$= \tan^{-1}\left(\frac{x+1}{4}\right) + c$$

(25) Let $I = \int_0^3 x\sqrt{x+1} \, dx$

Let $u = x+1$, $\therefore du = dx$ & $u - 1 = x$

Also, when $x = 0$, $u = 1$ & $x = 3$, $u = 4$

So $$I = \int_1^4 (u-1) \cdot u^{1/2} \, du = \int_1^4 u^{3/2} - u^{1/2} \, du$$

$$= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4$$

(26) let $I = \int_1^2 (x+2)(x-1)^5 \, dx$

Now let $u = x-1$, $\therefore du = dx$ & $u+3 = x+2$

Also, if $x=1$, $u=0$
& $x=2$, $u=1$

So $I = \int_0^1 (u+3)\, u^5 \, du = \int_0^1 u^6 + 3u^5 \, du$

$$= \left[\frac{u^7}{7} + 3\frac{u^6}{6} \right]_0^1 = \frac{1}{7} + \frac{1}{2} = \frac{9}{14}$$

(27) let $I = \int_1^2 x^2 (x-1)^5 \, dx$

Now let $u = x-1$, so $du = dx$ & $(u+1)^2 = x^2$

Also when $x=1$, $u=0$
& $x=2$, $u=1$

So $I = \int_0^1 (u+1)^2 \cdot u \, du = \int_0^1 u^3 + 2u^2 + u \, du$

$$= \left[\frac{u^4}{4} + 2\frac{u^3}{3} + \frac{u^2}{2} \right]_0^1 = 1\frac{5}{12}$$

(28) Let $I = \int_1^2 x(2x-3)^4\,dx$

Now let $u = 2x-3 \Rightarrow \frac{1}{2}du = dx$ & $\frac{u+3}{2} = x$

Also let $x=1$, $\therefore u=-1$
& $x=2$, $\therefore u=1$

So $I = \int_{-1}^{1} \frac{1}{2}\left(\frac{u+3}{2}\right)\cdot u^4\,du = \frac{1}{2}\int_{-1}^{1} \frac{1}{2}u^5 + \frac{3}{2}u^4\,du$

$= \left[\frac{1}{4}\cdot\frac{1}{6}u^6 + \frac{3}{4}\cdot\frac{1}{5}u^5\right]_{-1}^{1} = \left(\frac{1}{24}+\frac{3}{20}\right) - \left(\frac{1}{24}-\frac{3}{20}\right)$

$= \frac{3}{10}$

(29) Let $I = \int_0^1 4x\cdot(2x-1)^4\,dx$

Now let $u = 2x-1$, so $\frac{1}{2}du = dx$ & $4\frac{(u+1)}{2} = 4x$

Also if $x=0$, $u=-1$
& $x=1$, $u=1$

So $I = \int_{-1}^{1} \frac{1}{2}\cdot 2(u+1)\cdot u^4\,du = \int_{-1}^{1} u^5 + u^4\,du$

$= \left[\frac{u^6}{6}+\frac{u^5}{5}\right]_{-1}^{1} = \left(\frac{1}{6}+\frac{1}{5}\right) - \left(\frac{1}{6}+\frac{1}{5}\right) = \frac{2}{5}$

(30) let $I = \int_0^{1/2} \frac{1}{1+4x^2}\, dx$

Now let $x = \frac{1}{2}\tan\theta \Rightarrow x^2 = \frac{1}{4}\tan^2\theta$ & $dx = \frac{1}{2}\sec^2\theta\, d\theta$

Also if $x = 0$, $\theta = 0$
& $x = \frac{1}{2}$, $\theta = \frac{\pi}{4}$

So $I = \int_0^{\pi/4} \frac{1}{1+4\left(\frac{1}{4}\tan^2\theta\right)} \cdot \frac{1}{2}\sec^2\theta\, d\theta$

$= \int_0^{\pi/4} \frac{1}{\sec^2\theta} \cdot \frac{1}{2}\sec^2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/4} 1\, d\theta = \frac{1}{2}\theta\Big|_0^{\pi/4} = \frac{\pi}{8}$

(31) let $I = \int_0^{3/2} \sqrt{9-4x^2}\, dx$

Now let $x = \frac{3}{2}\sin\theta \Rightarrow x^2 = \frac{9}{4}\sin^2\theta$ & $dx = \frac{3}{2}\cos\theta$

Also if $x = 0$, $\theta = 0$
& if $x = \frac{3}{2}$, $\theta = \frac{\pi}{2}$

$\therefore$

$I = \int_0^{\pi/2} \left(9 - 4\cdot\frac{9}{4}\sin^2\theta\right)^{1/2} \cdot \frac{3}{2}\cos\theta\, d\theta$

$= \int_0^{\pi/2} \sqrt{9\cos^2\theta} \cdot \frac{3}{2}\cos\theta\, d\theta$

So $\quad I = \frac{9}{2}\int_0^{\pi/2} \cos^2\theta \, d\theta$

$$= \frac{9}{2}\int_0^{\pi/2} \frac{\cos 2\theta + 1}{2} \, d\theta$$

$$= \frac{9}{4}\left(\frac{1}{2}\sin 2\theta + \theta\right)\Big|_0^{\pi/2} = \frac{9}{4}\left[\left(0 + \frac{\pi}{2}\right) - 0\right]$$

$$= \frac{9\pi}{8}$$

(32) let $\quad I = \int_0^{\pi/2} \frac{3}{1+\sin\theta} \, d\theta \quad$ where $t = \tan\frac{\theta}{2}$

$\therefore \quad \sin\theta = \frac{2t}{1+t^2} \quad , \quad \therefore \quad \cos\theta \cdot \frac{d\theta}{dt} = 2 \cdot \frac{1-t^2}{(1+t^2)^2}$

Since $\cos\theta = \frac{1-t^2}{1+t^2}$ we have $\quad \frac{1-t^2}{1+t^2} \cdot d\theta = 2 \cdot \frac{1-t^2}{(1+t^2)^2} dt$

$$\therefore \quad d\theta = 2 \cdot \frac{(1+t^2)(1-t^2)}{(1+t^2)^2(1-t^2)} dt$$

$$= \frac{2}{1+t^2} dt$$

Also when $\theta = 0, \; t = 0$

$\qquad \theta = \frac{\pi}{2}, \; t = 1$

So $\quad I = \int_0^1 \frac{3}{1 + \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt$

$$\therefore \quad I = 6\int_0^1 \frac{1+t^2}{1+t^2+2t} \cdot \frac{1}{1+t^2}\, dt$$

$$= 6\int_0^1 \frac{1}{(1+t)^2}\, dt = -6\,(1+t)^{-1}\Big|_0^1$$

$$= -6\left(2^{-1} - 1^{-1}\right)$$

$$= -3 + 6 = 3$$

(33) let $I = \displaystyle\int_0^{2\pi/3} \frac{3}{5+4\cos\theta}\, d\theta$, where $t = \tan\frac{\theta}{2}$ $\circledast$

So $\cos\theta = \dfrac{1-t^2}{1+t^2}$ & $-\sin\theta \dfrac{d\theta}{dt} = -\dfrac{4t}{(1+t^2)^2}$

Since $\sin\theta = \dfrac{2t}{1+t^2}$ we have $-\dfrac{2t}{1+t^2} \cdot d\theta = \dfrac{-4t}{(1+t^2)^2}\, dt$

$\therefore \quad d\theta = \dfrac{2}{1+t^2}\, dt$

Also when $\theta = 0$, $t = 0$.
& " $\theta = \frac{2\pi}{3}$, $t = \pm\sqrt{3}$

Now if $t = +\sqrt{3}$ Then by $\circledast$ $\sqrt{3} = \tan\frac{\theta}{2} \Rightarrow \theta = \frac{2\pi}{3}$

if $t = -\sqrt{3}$ Then by $\circledast$ $-\sqrt{3} = \tan\frac{\theta}{2} \Rightarrow \theta = -\frac{2\pi}{3}$ which is not a given limit of I. So use only $t = +\sqrt{3}$.

Hence $\quad I = \displaystyle\int_0^{\sqrt{3}} \frac{3}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\, dt$

$$= 6\int_0^{\sqrt{3}} \frac{1+t^2}{5+5t^2+4-4t^2} \cdot \frac{1}{1+t^2}\, dt$$

$$= 6\int_0^{\sqrt{3}} \frac{1}{9+t^2}\, dt$$

Now let $t = 3\tan x$, $\therefore\ dt = 3\sec^2 x\, dx$

$t = \sqrt{3} \Rightarrow x = \pi/6$
$t = 0 \Rightarrow x = 0$

So $\quad I = 6\displaystyle\int_0^{\pi/6} \frac{1}{9+9\tan^2 x} \cdot 3\sec^2 x\, dx$

$$= 18\int_0^{\pi/6} \frac{\sec^2 x}{9\sec^2 x}\, dx = 2\int_0^{\pi/6} 1\, dx$$

$$= 2x\Big|_0^{\pi/6} = \pi/3.$$

(34) Let $I = \int_{-\pi/2}^{\pi/2} \frac{3}{4+5\cos\theta}\, d\theta$ with $t = \tan\frac{\theta}{2}$

Now, $\cos\theta = \frac{1-t^2}{1+t^2}$ so $-\sin\theta\, d\theta = -\frac{4t}{(1+t^2)^2}\, dt$

Since $\sin\theta = \frac{2t}{1+t^2}$ we have $\frac{-2t}{1+t^2}\, d\theta = \frac{-4t}{(1+t^2)^2}\, dt$

$\therefore\ d\theta = \frac{2}{1+t^2}\, dt$

Also when $\theta = -\frac{\pi}{2}$, $t = 1$

when $\theta = -\pi/2$, $t = -1$

So $I = \int_{-1}^{1} \frac{3}{4+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\, dt$

$= \int_{-1}^{1} \frac{6(1+t^2)}{4+4t^2+5-5t^2} \cdot \frac{1}{1+t^2}\, dt$

$= \int_{-1}^{1} \frac{6}{9-t^2}\, dt = 6\int_{-1}^{1} \frac{1}{(3+t)(3-t)}\, dt$

use partial fractions: $\frac{1}{(3+t)(3-t)} = \frac{A}{3+t} + \frac{B}{3-t}$

So $1 = A(3-t) + B(3+t)$

when $t=3$: $1 = 6B \Rightarrow B = 1/6$

$t=-3$: $1 = 6A \Rightarrow A = 1/6$

So $I = 6\int_{-1}^{1} \frac{1/6}{3+t} + \frac{1/6}{3-t}\, dt$

$$= 6\left[\frac{1}{6}\ln|3+t| - \frac{1}{6}\ln|3-t|\right]_{-1}^{1}$$

Since $3 \pm t$ is positive in $-1 \le t \le 1$ we can write ln without modulus signs.

$$\therefore\ I = \ln\frac{(3+t)}{(3-t)}\bigg|_{-1}^{1} = \ln\frac{4}{2} - \ln\frac{2}{4} = \ln 4$$

(35) let $I = \int_0^{\pi/2} \frac{5}{3\sin\theta + 4\cos\theta}\, d\theta$ with $t = \tan\frac{\theta}{2}$

Since $\sin\theta = \frac{2t}{1+t^2}$ & $\cos\theta = \frac{1-t^2}{1+t^2}$ ⊛

Then $d\theta = \frac{2}{1+t^2}\, dt$ Via either of the two Equations above

(See Ex 32 & 33 for derivation)

Also $\left.\begin{array}{l} \text{if } \theta = 0, \quad t = 0 \\ \text{if } \theta = \pi/2, \quad t = 1 \end{array}\right\}$ by either of ⊛

(if you also get a "$t=-1$" value, Note That This is Not valid Since it does Not give a 0 or $\pi/2$ limit on I)

So $I = \int_0^1 \frac{5}{3\cdot\frac{2t}{1+t^2} + 4\cdot\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$

$$\therefore \quad I = 10\int_0^1 \frac{1+t^2}{6t+4-4t^2}\cdot\frac{1}{1+t^2}\,dt$$

$$= \frac{-10}{2}\int_0^1 \frac{1}{2t^2-3t-2}\,dt = -5\int_0^1 \frac{1}{(2t+1)(t-2)}\,dt$$

By Partial fractions: $\dfrac{1}{(2t+1)(t-2)} = \dfrac{A}{2t+1} + \dfrac{B}{t-2}$

So $\quad 1 = A(t-2) + B(2t+1)$

When $t=2$: $\quad 1 = 5B \Rightarrow B = 1/5$

when $t=-\frac{1}{2}$: $\quad 1 = -\frac{5}{2}A \Rightarrow A = -\frac{2}{5}$

So $\quad I = -5\int_0^1 \dfrac{-2/5}{2t+1} + \dfrac{1/5}{t-2}\,dt$

$$= -5\left[-\frac{2}{5}\cdot\frac{1}{2}\ln|2t+1| + \frac{1}{5}\ln|t-2|\right]_0^1$$

$$= \Big[\ln|2t+1| - \ln|t-2|\Big]_0^1 =$$

$$= \left(\ln|3| - \ln|-1|\right) - \left(\ln|1| - \ln|-2|\right)$$

$$= \ln 3 + \ln 2 = \ln 6$$